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# Finite Element Procedures

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**Second Edition**

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**— Finite element analysis is an art to predict the future —**

***To my students***

*... Progress in design of new structures seems to be unlimited.*

Last sentence of article: "The Use of the Electronic Computer in Structural Analysis," by K. J. Bathe (undergraduate student), published in *Impact, Journal of the University of Cape Town Engineering Society*, pp. 57 – 61, 1967.

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# Preface

Finite element procedures are now an important and frequently indispensable part of engineering analyses and scientific studies. Finite element computer programs are now widely used in practically all branches of engineering and the sciences for the analysis of structures, solids, fluids and multi-physics problems.

## The philosophy of the book

The first edition of this book was published in 1996 and since then the book has been printed over twenty times. For almost two decades I did not update the book, but then decided that small updates and improvements which are in the philosophy of the book should be incorporated without increasing the size of the book.

The basic philosophy of the book is that *the text shall not present a survey of finite element methods*. For such an endeavor, a number of volumes would be needed. Instead, the book concentrates only on basic finite element procedures, very useful in engineering and the sciences, and that will probably be employed for many years to come.

An important aspect of a finite element procedure is its reliability, so that the method can be used in a confident manner. Hence *this book focuses on finite element procedures that are general, reliable and efficient for engineering analyses and scientific studies*.

A deep understanding of finite element procedures is achieved only if sufficient attention is given to both the physical and mathematical characteristics of the procedures. The combined physical and mathematical understanding greatly enriches the confident use and further development of finite element methods and is therefore emphasized in this text.

Although many papers on research and development of finite element methods have been published during the recent decades, the basic formulations and procedures, focused upon in the 1996 book, hardly changed. For this reason, and because the 1996 edition was well received, I did not attempt to improve upon most of the 1996 text, instead, the reader will see that only some important recent research work is now briefly mentioned and some recently developed novel procedures (that are in the philosophy of the book) have replaced earlier text. The changes are not large, but of course, I had to call this book a second edition.

While I am very pleased to have written the book, the work on the book required an immense effort on my part over many years.

## Acknowledgments

I am very grateful that the Mechanical Engineering Department of M.I.T. has provided me, for over 40 years, with an excellent environment in which to pursue my interests in teaching, research, and scholarly writing.

A text of significant depth and breadth on a subject that came to life only in the 1960s and that has experienced tremendous advances can be written only by an author who has had the benefit of interacting with many people in the field.

I have been truly fortunate to work with many outstanding students at M.I.T., for which I am very thankful. It has been a great privilege to be their teacher and work with them. Of much value has also been that I have been intimately involved at my company ADINA R & D, Inc. in the development of finite element methods for industry.

I would like to thank all my students, colleagues and friends who contributed to my knowledge and understanding of finite element methods. My interaction with them, given in papers referred to in the book, has resulted in achievements that I am very proud of, see also my book "To Enrich Life".

The typesetting of the new sections and the printing of the book was accomplished through the help of ADINA R & D, and I would like to particularly thank Victor Lee of the company for his help.

## Closure

I humbly hope that this book will be of value to many students and professionals to increase their understanding of finite element procedures. In that sense, I would like to close this Preface by quoting Shakespeare:

*"... knowledge is the wing wherewith we fly to heaven."*

*Klaus-Jürgen Bathe  
M.I.T.*